H2 MATH 9758 SPECIMEN PAPER 1

1 A circular ink-blot is expanding such that the rate of change of its diameter D with respect to time t is 0.25 cm/s. Find the rate of change of both the circumference and the area of the circle with respect to t when the radius of the circle is 1.5 cm. Give your answers correct to 4 decimal places. [4]

2 The curve C with equation $y = x^3$ is transformed onto the curve with equation $y = \mathrm{f}(x)$ by a translation of 2 units in the negative x-direction, followed by a stretch of factor $\frac{1}{2}$ parallel to the y-axis, followed by a translation of 1 unit in the positive y-direction.

(i) Write down the equation of the new curve. [1]

(ii) Sketch C and the curve with equation $y = \mathrm{f}(x)$ on the same diagram, stating the exact values of the coordinates of the points where $y = \mathrm{f}(x)$ crosses the x- and y-axes. Find the x-coordinate(s) of the point(s) where the two curves intersect, giving your answer(s) correct to 3 decimal places. [4]

3 (i) Sketch the curve with equation $y = \dfrac{x^2 - 12}{x}$, giving the exact coordinates of the point(s) where the curve crosses the axes and the equations of any asymptotes. [4]

(ii) Hence, or otherwise, solve the inequality $\dfrac{x^2 - 12}{x} < 1$. [3]

4 A science student is investigating the elasticity of a new compound. She drops a ball made of the new compound vertically onto a hard surface and measures the height reached by the ball after each successive bounce. She drops the ball from an initial height of 200 cm and she estimates that the height the ball reaches after each bounce is $\frac{8}{9}$ of the height reached by the previous bounce.

(i) Find the total distance that the ball has travelled when it reaches the highest point after the fourth bounce. Give your answer correct to the nearest centimetre. [2]

(ii) The ball is considered to have stopped bouncing when a bounce first results in the height the ball reaches being less than 0.01 cm. Find how many bounces the ball has made and the total distance that the ball has travelled in this case. Give your answer correct to the nearest centimetre. [6]

5 The curve C has equation $y = \frac{1}{x}(\ln x)^3$, where $x > 1$.

(i) Find the exact x-coordinate, $x = x_1$, of the turning point on C and explain whether it is a maximum or a minimum turning point. [4]

(ii) Without using a calculator, find the exact area of the region between C, the x-axis and the lines with equations $x = \mathrm{e}$ and $x = x_1$. [3]

6 (a) The non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$. Given that $\mathbf{b} \neq -\mathbf{c}$, find a linear relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [3]

(b) The variable vector $\mathbf{v}$ satisfies the equation $\mathbf{v} \times (\mathbf{i} - 3\mathbf{k}) = 2\mathbf{j}$. Find the set of vectors $\mathbf{v}$ and describe this set geometrically. [5]

7 **Do not use a calculator in answering this question.**

(a) Showing your working, find the complex numbers z and w which satisfy the simultaneous equations

$$2\mathrm{i}z + (1 - 2\mathrm{i})w = 4 \text{ and}$$

$$(1 + \mathrm{i})z + (2 + \mathrm{i})w = 3 \ .$$ [6]

(b) The complex number u is given by $u = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \pi$. Show that $1 - u^2 = -2\mathrm{i}u\sin\theta$ and hence or otherwise find the modulus and argument of $1 - u^2$ in terms of θ. [5]

8 The astroid, a curve C which is used to characterise various properties of energy and magnetism, has parametric equations

$$x = a\cos^3 t, \ y = a\sin^3 t,$$

where $0 \leqslant t \leqslant \frac{1}{2}\pi$ and a is a positive constant.

(i) Find the equation of the tangent to C at the point P with parameter p. [3]

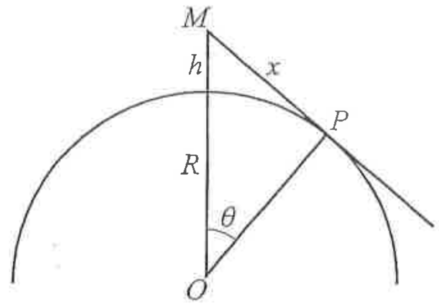
(ii) The tangent at P meets the x-axis at the point A and meets the y-axis at the point B. Show that the length AB depends only on a. [3]

It is given that $a = 1$.

(iii) Find a cartesian equation of C. [2]

(iv) The region bounded by C and the x- and y-axes is rotated through 360° about the y-axis. Find the exact value of the volume of revolution of the solid formed. [4]

9



A man M is at the top of a mountain which is of height h km. The radius of the earth is assumed to be a constant R km. The furthest point on the earth's surface that the man can see is a point P such that $MP = x$ km and the angle $POM = \theta$, where O is the centre of the earth (see diagram). You may assume that the height of the man is negligible.

(i) Show that $x = (2hR)^{\frac{1}{2}}\left(1 + \frac{h}{2R}\right)^{\frac{1}{2}}$. [3]

(ii) It is given that h is small compared to R. Show that, if $\alpha = \frac{h}{R}$, $\sin\theta \approx (2\alpha)^{\frac{1}{2}}\left(1 - \frac{3}{4}\alpha\right)$. [5]

(iii) The man M has a scientific instrument which enables him to estimate the angle between PM and the horizontal. Given that this angle is 2° and that the radius of the earth is 6375 km, find estimates for the values of α and h. [4]

10 The point A has coordinates $(-1, 2, -1)$. The line l has equation $\frac{x}{2} = \frac{y+1}{-3} = \frac{z-2}{1}$.

(i) Find the cartesian equation of the plane π which contains A and is perpendicular to l. [3]

(ii) Hence, or otherwise, find the coordinates of the point P on l which is closest to A. [3]

(iii) The line m passes through the point with coordinates $(4, -5, 10)$ and P. The line n lies in the same plane as l and m. Find a cartesian equation for n if n is the reflection of the line m about the line l. [6]

11 A pond has a surface area of $10\,\text{m}^2$. Biologists have planted an area of new weeds. They estimate how many weeds there are and the rate at which they are spreading by finding the area of the pond the weeds cover at various times. They believe that the area, $A\,\text{m}^2$, of weeds present at time t months is such that the rate at which the area is increasing is proportional to the product of the area of pond covered by the weeds and the area of the pond not covered by the weeds. It is known that the initial area of weeds is $2\,\text{m}^2$ and that the area of weeds is $4\,\text{m}^2$ after 5 months.

(i) Write down a differential equation expressing the relation between A and t. Find the time at which 80% of the pond is covered in weeds, giving your answer correct to 2 decimal places. [8]

(ii) Given that the experiment is stopped after 2 years, find the area of pond covered by weeds, giving your answer correct to 2 decimal places. [2]

(iii) Write the solution of the differential equation in the form $A = \text{f}(t)$ and sketch this curve. [4]

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1 **(i)** The function f is defined as follows:

$$f : x \mapsto 3\cos x - 2\sin x,\ x \in \mathbb{R},\ -\pi \leqslant x < \pi.$$

Write f(x) as $R\cos(x+\alpha)$, where R and α are constants to be found. Hence, or otherwise, find the range of f and sketch the curve. [4]

(ii) The function g is defined as follows:

$$g : x \mapsto 3\cos x - 2\sin x,\ x \in \mathbb{R},\ -\alpha \leqslant x \leqslant b.$$

Given that the function g^{-1} exists, write down the largest value of b. Find $g^{-1}(x)$. [3]

2 The first four terms of a sequence of numbers are 3, 1, 1 and 3. S_n is the sum of the first n terms of this sequence.

(i) Explain why S_n cannot be a quadratic polynomial in n. [2]

It is given that S_n is a cubic polynomial.

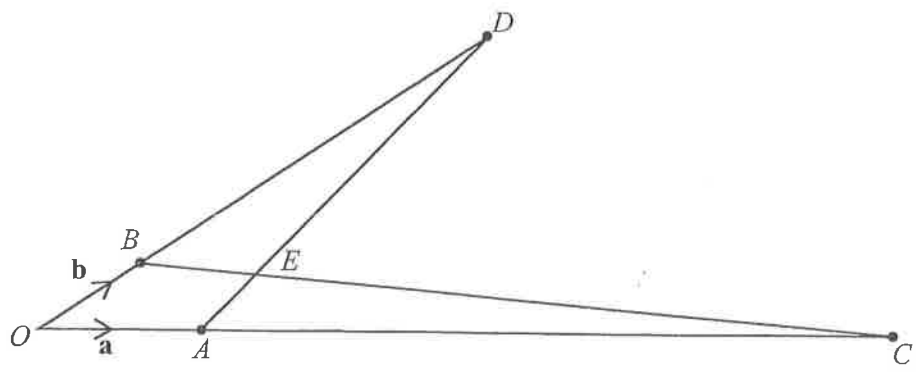
(ii) Find S_n in terms of n. [4]

(iii) Find an expression in terms of n for the nth term of the sequence. [3]

3 **(a)** The angle between the vectors $3\mathbf{i} - 2\mathbf{j}$ and $6\mathbf{i} + d\mathbf{j} - \sqrt{7}\mathbf{k}$ is $\cos^{-1}\left(\frac{6}{13}\right)$.

Show that $2d^2 - 117d + 333 = 0$. [3]

(b)



With reference to origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AC} = 5\mathbf{a}$ and $\overrightarrow{BD} = 3\mathbf{b}$. The lines AD and BC cross at E (see diagram).

(i) Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [6]

(ii) The point F divides the line CD in the ratio 5 : 3. Show that O, E and F are collinear, and find $OE : OF$. [4]

4 **(i)** Given that $y = \tan(e^{2x} - 1)$, show that $\frac{dy}{dx} = ke^{2x}(1 + y^2)$, where k is to be found. Hence find the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ when $x = 0$. [6]

(ii) Write down the first three non-zero terms in the Maclaurin series for $\tan(e^{2x} - 1)$. [1]

(iii) The first two non-zero terms in the Maclaurin series for $\tan(e^{2x} - 1)$ are equal to the first two non-zero terms in the series expansion of $e^{ax}\ln(1 + nx)$. By using appropriate expansions from the List of Formulae (**MF26**), find the constants a and n. Hence find the third non-zero term of the series expansion of $e^{ax}\ln(1 + nx)$ for these values of a and n. [4]

Section B: Probability and Statistics [60 marks]

5 This question is about six couples. Each couple consists of a husband and a wife.

The 12 people visit a theatre, and sit in a row of 12 seats.

(i) In how many different ways can the 12 people sit so that each husband and wife in a couple sit next to each other? [2]

(ii) In how many different ways can the 12 people sit so that the 6 wives all sit next to each other, and none of the wives sits next to her own husband? [3]

The group decides to form a committee to arrange future outings. The committee will consist of 3 of the 12 people. At least 1 of the wives will be on the committee but no husband and wife couple will be included.

(iii) In how many ways can the committee be formed? [3]

6 Giant pumpkins are often irregular in shape. In order to account for the different shapes of pumpkins, growers of giant pumpkins measure the size of a pumpkin by a combination of three measurements, called the 'over the top' length. Pumpkin growers keep records so that they can estimate the mass of giant pumpkins while they are still growing. The over the top lengths (d m) and the masses (m kg) of a random sample of 7 giant pumpkins are as follows.

d	2.31	2.9	4.05	5.5	6.7	7.92	9.17
m	11	14	47	104	170	282	449

(i) Draw a scatter diagram of these data, and explain how you know from your diagram that the relationship between m and d should not be modelled by an equation of the form $y = ax + b$. [2]

(ii) Which of the formulae $m = ed^2 + f$ and $m = gd^3 + h$, where e, f, g and h are constants, is the better model for the relationship between m and d? Explain fully how you decided, and find the constants for the better formula. [5]

(iii) Use the formula you chose from part **(ii)** to estimate the mass of a giant pumpkin with

(a) over the top length 6 m,
(b) over the top length 12 m.

Explain which of your two estimates is more reliable. [3]

7 'Bings' are sweets that are sold in packets of 6. Each packet is made up of randomly chosen coloured sweets. On average 10% of Bings are yellow.

(i) Explain why a binomial distribution is appropriate for modelling the number of yellow sweets in a packet. Find the probability that a randomly chosen packet of Bings contains no more than one yellow sweet. [3]

(ii) Kev buys 90 randomly chosen packets of Bings. Find the probability that at least 80 of these packets contain no more than one yellow sweet. [2]

On average the proportion of Bings that are red is p. It is known that the modal number of red sweets in a packet is 2.

(iii) Use this information to find exactly the range of values that p can take. [4]

8 A bag contains 3 blue counters, 1 red counter and y yellow counters. Darvina chooses 3 counters at random from the bag, without replacement. The random variable S is the sum of the number of blue counters chosen and **twice** the number of red counters chosen.

(i) Show that $\mathrm{P}(S=3)=\dfrac{6(3y+1)}{(y+4)(y+3)(y+2)}$. [2]

(ii) Given that $\mathrm{P}(S=3)=\dfrac{7}{20}$, calculate y. Hence find the probability distribution of S. [6]

9 A type of metal bolt is manufactured with a nominal radius of 0.8 cm. In fact, the radii of the bolts, measured in cm, have the distribution $\mathrm{N}(0.8, 0.01^2)$.

(i) Find the percentage of bolts that have a radius between 0.79 cm and 0.82 cm. [1]

Metal washers are manufactured to fit on the bolts. The inside radii of the washers, measured in cm, have the distribution $\mathrm{N}(0.81, 0.012^2)$.

(ii) Write down the distribution of the inside circumference of the washers, in cm, and find the circumference that is exceeded by 5% of the washers. [4]

A bolt and a washer are a 'good fit' if

- the inside radius of the washer is greater than the radius of the bolt and
- the inside radius of the washer is not more than 0.04 cm greater than the radius of the bolt.

(iii) A washer is chosen at random, and a bolt is chosen at random. Find the probability that the washer and bolt are a good fit. [3]

The outside radii of the washers, measured in cm, have the distribution $\mathrm{N}(\mu, \sigma^2)$. It is known that 15% of the washers have an outside radius greater than 1.25 cm and 25% have an outside radius of less than 1.15 cm.

(iv) Find the values of μ and σ. [4]

10 The average time required for the manufacture of a certain type of electronic control panel is 17 hours. An alternative manufacturing process is trialled, and the time taken, t hours, for the manufacture of each of 50 randomly chosen control panels using the alternative process is recorded. The results are summarised as follows.

$$n = 50 \qquad \Sigma t = 835.7 \qquad \Sigma t^2 = 14067.17$$

The Production Manager wishes to test whether the average time taken for the manufacture of a control panel is different using the alternative process, by carrying out a hypothesis test.

(i) Explain whether the Production Manager should use a 1-tail test or a 2-tail test. [1]

(ii) Explain why the Production Manager is able to carry out a hypothesis test without knowing anything about the distribution of the times taken to manufacture the control panels. [2]

(iii) Find unbiased estimates of the population mean and variance and carry out the test at the 10% level of significance for the Production Manager. [6]

(iv) Suggest a reason why the Production Manager might be prepared to use an alternative process that takes a longer average time than the original process. [1]

The Finance Manager wishes to test whether the average time taken for the manufacture of a control panel is **shorter** using the alternative process. The Finance Manager finds that the average time taken for the manufacture of each of 40 randomly chosen control panels, using the alternative process, is 16.7 hours. He carries out a hypothesis test at the 10% level of significance.

(v) Explain, with justification, how the population variance of the times will affect the conclusion made by the Finance Manager. [3]

Paper 1 Answers

1) 0.7854 cm/s; 1.1781 cm²/s **2**(i) $y = \frac{1}{2}(x+2)^3 + 1$ **3**(ii) $x < -3$ or $0 < x < 4$ **4**(i) 1277 (ii) 85; 3400 cm

5(i) e^3 (ii) 20 unit² **6**(a) $\lambda \mathbf{a} = \mathbf{b} + \mathbf{c}, \lambda \in \mathbb{R}$ (b) $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, t \in \mathbb{R}$ **7**(a) $z = \frac{1}{2}(1-3i); w = \frac{1}{5}(3+i)$

7(b) $2\sin\theta$; $-\frac{\pi}{2} + \theta$ **8**(i) $y\cos p + x\sin p = a\sin p\cos p$ (iii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ (iv) $\frac{16}{105}\pi$

9(iii)0.00061; 3883 m **10**(i) $2x - 3y + z = -9$ (ii) $(-2, 2, 1)$ (iii) $\frac{x+2}{6} = \frac{y-2}{-11} = \frac{z-1}{-3}$

11(i) $\frac{dA}{dt} = kA(10-A)$; 14.13 (ii) 9.65 (iii) $A = \frac{10e^{pt}}{4+e^{pt}}$ where $p = \frac{1}{5}\ln\frac{8}{3}$

Paper 2 Answers

1(i) $R = \sqrt{3}, \alpha = \tan^{-1}\frac{2}{3}$; $\left[-\sqrt{13}, \sqrt{13}\right]$ (ii) largest $b = \pi - \alpha$; $g^{-1}(x) = \cos^{-1}\frac{x}{\sqrt{13}} - \alpha$

2(ii) $\frac{1}{3}n^3 - 2n^2 + \frac{14}{3}n$ (iii) $n^2 - 5n + 7$ **3**(b)(i) $\frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}$ (ii) 8 : 23

4(i) $k = 2$; 2, 4, 24 (ii) $\tan\left(e^{2x} - 1\right) = 2x + 2x^2 + 4x^3 + \ldots$ (iii) $a = 2, n = 2$; $\frac{8}{3}$ **5**(i) 46080 (ii) 2678400 (iii) 140

6(ii) $m = 0.572d^3 + 3.74$ (iii)(a) 127 (b) 992 **7**(i)0.886 (ii) 0.549 (iii) $\frac{2}{7} < p < \frac{3}{7}$ **8**(ii) $y = 2$

s	1	2	3	4
$P(S = s)$	18/120	42/120	42/120	18/120

9(i) 81.9% (ii) 5.21 cm (iii) 0.712 (iv) 1.1894; 0.0585 **10**(iii) 16.714; 2.03; p-value = 0.155 (do not reject H_0)